# Quantum mechanics II, Problems 8- Perturbation Theory II

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## Problem 1 : Degenerate Perturbation Theory for a 3-State System

We consider the following Hamiltonian acting on a spin 1 :

$$\hat{H} = -D\hat{S}_z^2 + \lambda B\hat{S}_x \tag{1}$$

This is a model that can be realistic in certain materials. The first term represents an anisotropy, and the second a magnetic field along the $x$ direction. We propose to diagonalize this Hamiltonian by considering the term $\lambda B\hat{S}_x$ as a perturbation. Subsequently, we assume that $B$ and $D$ are non-zero. *Reminder from "quantum physics 1" course :* For a spins $s$, we have $\hat{S}_z |s,m\rangle = \hbar m |s,m\rangle$ with $-s \leq m \leq s$. And the raising/lowering operators $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$ acting on these eigenvectors gives $\hat{S}_\pm |s,m\rangle = \sqrt{s(s+1) - m(m\pm 1)}\, |s, m\pm 1\rangle$.

1. Under what condition does the Hamiltonian commute with $\hat{S}_z$ ? In this case, give the eigenvalues and eigenvectors of $\hat{H}$.
2. Subsequently, we consider $\lambda \neq 0$. Write down the matrix of the Hamiltonian in the basis of eigenstates of $\hat{S}_z$. Using second order perturbation theory, compute the energy correction for the state $|m = 0\rangle$. Calculate the correction to the associated eigenvector, to first order in perturbation theory.
3. To calculate the effect of the perturbation on the other two states, it is necessary to use degenerate perturbation theory. What is the matrix of the operator $\hat{S}_x$ in the degenerate subspace ? Deduce that the first-order correction is zero.

## Problem 2 : Hydrogen atom in an external magnetic field

The Hamiltonian of a hydrogen atom under a weak uniform magnetic field directed along the $z$ axis can be expressed as

$$H = \frac{(\mathbf{p} + e\mathbf{A})^2}{2m} - \frac{e^2}{|\mathbf{r}|} - \boldsymbol{\mu} \cdot \mathbf{B} \ . \tag{2}$$

Here $\boldsymbol{\mu} = 2\mu_e \mathbf{s}$, with $\mu_e$ the electron magnetic moment and $\mathbf{s}$ the electron spin. Using the radial gauge $\mathbf{A} = \frac{1}{2}(-By, Bx, 0) = \frac{1}{2}\mathbf{B} \times \mathbf{r}$, and neglecting, for small $B$, terms of order $B^2$, the Hamiltonian can then be rewritten as

$$\begin{aligned} H &= \frac{\mathbf{p}^2}{2m} - \frac{e^2}{|\mathbf{r}|} + \frac{e}{2m}(\mathbf{p} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{p}) - 2\mu_e \mathbf{s} \cdot \mathbf{B} \\ &= H_0 + \frac{e}{4m}(\mathbf{p} \cdot (\mathbf{B} \times \mathbf{r}) + (\mathbf{B} \times \mathbf{r}) \cdot \mathbf{p}) - 2\mu_e \mathbf{s} \cdot \mathbf{B} \\ &= H_0 + \frac{e\hbar}{2m}\mathbf{B} \cdot \boldsymbol{\ell} - 2\mu_e \mathbf{s} \cdot \mathbf{B} \end{aligned} \tag{3}$$

where $H_0 = H|_{\boldsymbol{B}=0}$ Calculate the splitting of the hydrogen energy levels to first order in $\mathbf{B}$.

*Hint.* By rotational invariance, the spectrum cannot depend on the direction of the magnetic field. This invariance can be used to simplify the calculations.

Problem 3 : Perturbed two-dimensional harmonic oscillator

Consider a two-dimensional harmonic oscillator governed by the Hamiltonian

$$H = H_0 + gH_1 \ , \qquad H_0 = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2) \ , \qquad H_1 = x^2y^2 \ . \tag{4}$$

The model, for example, can describe in an approximate way an optical phonon in a two dimensional lattice with square anisotropy.

1. Consider the unperturbed Hamiltonian $H_0$, which describes a two-dimensional harmonic oscillator. Describe the spectrum of $H_0$, expressing the eigenstates and the corresponding energies.
2. Study how the ground state, the first excited level, and the second excited level of the unpertubed Hamiltonian are modified by $H_1$, using perturbation theory to first order in $g$.

   *Hint.* It is convenient to express $x$ and $y$ in terms of the ladder operators for the harmonic oscillator.